DEPTH AND STANLEY DEPTH OF MULTIGRADED MODULES

ASIA RAUF

Abstract. We study the behavior of depth and Stanley depth along short exact sequences of multigraded modules and under reduction modulo an element.

Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ a polynomial ring in n variables over K. Let M be a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Let $m \in M$ be a homogeneous element in M and $Z \subseteq \{x_1, \dots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all elements mv, where v is a monomial in $K[Z]$. The multigraded K-subspace $mK[Z] \subset M$ is called Stanley space of dimension $|Z|$, if $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}:\ M = \bigoplus_{i=1}^r m_i K[Z_i]$. Set $\operatorname{sdepth} \mathcal{D} = \min\{|Z_i| : i = 1, \dots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) :\ \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M. In 1982, Richard P. Stanley [14, Conjecture 5.1] conjectured that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for all finitely generated $\mathbb{Z}^n$-graded S-modules M. The conjecture is discussed in some special cases in [7], [2], [9], [13], [1], [12] [4], [5].

If M is a finitely generated module over a Noetherian local ring (R, m) and $x \in m$ then it is well-known that $\dim M/xM \geq \dim M - 1$. Our Proposition 1.2 and Lemma 1.7, show that the above inequality is preserved for depth and sdepth when $M = S/I$ and $I \subset S$ is a monomial ideal and $x = x_k$ for any $k \in [n]$. If M is a general multigraded S-module, then we might have $\operatorname{depth} M/x_kM < \operatorname{depth} M - 1$ as shows Example 1.5. Also we might have $\operatorname{sdepth} M/x_kM < \operatorname{sdepth} M - 1$ even if x_k is regular on M, as shows Example 1.8.

As we know depth decreases by one if we reduce modulo a regular element. In [13, Theorem 1.1], it is proved that the corresponding statement holds for the Stanley depth in the case of $M = S/I$ where $I \subset S$ is a monomial ideal and f a monomial in S which is regular on M. The next question arises whether this is true for any multigraded module? The answer is no, see Example 1.8. Let

$$\mathcal{F}:\ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

be a chain of $\mathbb{Z}^n$-graded submodules of M. Then $\mathcal{F}$ is called a prime filtration of M if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and P_i is a monomial prime ideal for all i. We denote $\operatorname{Supp} \mathcal{F} = \{P_1, \dots, P_r\}$. A finitely generated module M is called almost clean if there exists a prime filtration $\mathcal{F}$ of M such that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$. We show in Lemma 1.9 that for almost clean module M and $x_k \in S$ being regular on M, we have $\operatorname{sdepth} M/x_kM \geq \operatorname{sdepth} M - 1$. However, we show in Proposition 1.10 that

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sdepth $M/x_kM \leq$ sdepth $M - 1$, if x_k is regular on M. As an application we get that Stanley's conjecture holds for M if it holds for the module M/x_kM (see Corollary 1.11). Moreover if M has a maximal regular sequence given by monomials then Stanley's conjecture holds for M (see Corollary 1.13).

Given a short exact sequence of finitely generated multigraded S-modules, then the Stanley depth of the middle one is greater than or equal to the minimum of Stanley depths of the ends (see Lemma 2.2). Several examples show that the "Depth lemma" is mainly wrong in the frame of sdepth (see Examples 2.5 and 2.6). However, we prove in Lemma 2.7 that if I is any monomial complete intersection of S, then sdepth I is greater than or equal to sdepth $S/I + 1$. But in general for any monomial ideal this inequality is still an open question.

In the last section, we prove that if $I \subset S_1 = K[x_1, \ldots, x_n]$, $J \subset S_2 = K[y_1, \ldots, y_m]$ are monomial ideals and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, then the Stanley depth of the tensor product of S_1/I and S_2/J (over K) is greater than or equal to the sum of sdepth S_1/I and sdepth S_2/J. This inequality could be strict as shows Example 3.2.

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1. The behavior of depth and sdepth under reduction modulo element

In dimension theory it is well known the following result (see e.g. [3, Proposition A.4], or [6, Corollary 10.9])

Theorem 1.1. *If (R, m) is a Noetherian local ring and M is finitely generated R-module, then for any $x \in m$ we have*

$$\dim M/xM \geq \dim M - 1.$$

If we consider reduction by a regular element, then the depth decreases by one. But what happens if we take reduction by a non-regular element?

Proposition 1.2. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, $I \subset S$ a monomial ideal and $R = S/I$. Then*

$$\operatorname{depth}(R/x_nR) \geq \operatorname{depth}(R) - 1. \tag{1}$$

Proof. Let $\bar{R} = R/x_nR \simeq S/(I, x_n)$. We denote $\bar{S} = K[x_1, \ldots, x_{n-1}]$ and let $x' = \{x_1, \ldots, x_{n-1}\}$, $x = \{x_1, \ldots, x_n\}$.

Let φ be the canonical map from R to $\bar{R}$ and α be the composite map

$$\bar{S} \longrightarrow S \longrightarrow R = S/I,$$

where the first map is the canonical embedding and the second map is the canonical surjection. It is clear that $\ker(\alpha) = I \cap \bar{S}$. Let α_1 be the composite map

$$\bar{S} \longrightarrow S \longrightarrow R = S/I \longrightarrow S/(I, x_n).$$

It is clear that α_1 is surjective. We claim that $\ker(\alpha_1) = I \cap \bar{S}$. One inclusion is obvious. To prove other inclusion, we consider a monomial $v \in \ker(\alpha_1)$, that is, $v \in (I, x_n)$. Since $v \in \bar{S}$ and I is a monomial ideal, it follows that $v \in I$. Let $\ker(\alpha_1) = \bar{I}$ then $\bar{S}/\bar{I} \simeq S/(I, x_n)$. It follows that the composition $\bar{R} \to R \to \bar{R}$ of the natural maps is the identity. Therefore, the S-module $\bar{R}$ is a direct summand of the S-module R. This implies that the S-module $H_i(x'; \bar{R})$ is a direct summand of $H_i(x'; R)$ for all i, where $H_i(x'; \bar{R})$ and $H_i(x'; R)$ are the i-th Koszul homology modules of x' with respect to $\bar{R}$ and R respectively. In particular, if $H_i(x'; \bar{R}) \neq 0$, then $H_i(x'; R) \neq 0$. Let $k = \max\{i \mid H_i(x' : \bar{R}) \neq 0\}$. Then depth $\bar{R} =$

$n-1-k$, by [3, Theorem 1.6.17]. Since $H_k(x';\bar{R}) \neq 0$, it follows that $H_k(x';R) \neq 0$ which implies that $H_k(x;R) \neq 0$ by [3, Lemma 1.6.18]. Therefore applying again [3, Theorem 1.6.17] it follows that $\operatorname{depth} R \leq n-k = \operatorname{depth} \bar{R}+1$. □

Corollary 1.3. *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{depth} S/(I:u) \geq \operatorname{depth} S/I$ *for all monomials $u \notin I$.*

Proof. Since $(I:uv) = ((I:u):v)$, where I is a monomial ideal and u and v are monomials, we may reduce to the case $u = x_n$, and apply recurrence. Then we have the exact sequence

$$0 \longrightarrow S/(I:x_n) \longrightarrow S/I \longrightarrow S/(I,x_n) \longrightarrow 0.$$

By Depth Lemma [15, Lemma 1.3.9] and Proposition 1.2, we obtain the required result. □

This Corollary does not hold (and so Proposition 1.2) if u is not a monomial, as we have the following example:

Example 1.4. Let $S = K[x,y,z,t]$ and $I = (x,y) \cap (y,z) \cap (z,t)$ and $u = y+z$. Then $J := (I:u) = (x,y) \cap (z,t)$ and $\operatorname{depth} S/J = 1 < 2 = \operatorname{depth} S/I$.

The Proposition 1.2 is not true in general. If M is a finitely generated graded R-module and $x \in R_1$ then we might have

$$\operatorname{depth}(M/xM) < \operatorname{depth}(M) - 1,$$

as shows the following example:

Example 1.5. Let $S = K[x,y,z,t]$, $M = (x,y,z)/(xt)$. We have $\operatorname{depth} M = 2$ and $M/xM = (x,y,z)/(x^2,xy,xz,xt)$. Since the maximal ideal is an associated prime ideal of M/xM, we get $\operatorname{depth} M/xM = 0$. Hence $\operatorname{depth}(M/xM) < \operatorname{depth}(M) - 1$.

In Proposition 1.2 we might have

$$\operatorname{depth}(R/x_nR) > \operatorname{depth}(R) - 1,$$

as shows the following example:

Example 1.6. Let $I = (x_1^2, x_1x_2, \ldots, x_1x_n) \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal of S and $R = S/I$. Then $\operatorname{depth}(R) = 0$ since the maximal ideal $(x_1,x_2,\ldots,x_n) \in \operatorname{Ass}(R)$. Since $R/x_1R = S/(x_1) \simeq K[x_2,\ldots,x_n]$, we get $\operatorname{depth}(R/x_1R) = n-1$. Hence $\operatorname{depth}(R/x_1R) > \operatorname{depth}(R) - 1$.

For the sdepth we have a statement similar to that of Proposition 1.2. Indeed in the proof of [13, Lemma 1.2] where it was shown that $\operatorname{sdepth} S/(I,x_n) = \operatorname{sdepth} S/I - 1$ if x_n is regular on S/I, we actually showed the following (without any assumption on x_n):

Lemma 1.7. *Let $S = K[x_1,\ldots,x_n]$ be a polynomial ring over the field K. Let $I \subset S$ be any monomial ideal. Then*

$$\operatorname{sdepth}(S/(I,x_n)) \geq \operatorname{sdepth}(S/I) - 1.$$

This lemma can not be extended to general multigraded modules M as shows the following example, where the variable is even regular on M.

Example 1.8. Let $M = (x,y,z)$ be an ideal of $S = K[x,y,z]$. Consider a Stanley decomposition $M = zK[x,z] \oplus xK[x,y] \oplus yK[y,z] \oplus xyzK[x,y,z]$. Since $\operatorname{sdepth} M \leq \dim S = 3$ and M is not a principle ideal, it follows $\operatorname{sdepth} M = 2$. Note that x induces a

non-zero element in the socle of M/xM which cannot be contained in any Stanley space of dimension greater or equal with one. Hence $\operatorname{sdepth} M/xM = 0$. Thus $\operatorname{sdepth} M/xM < \operatorname{sdepth} M - 1$.

However for the special case when M is *almost clean* (see [9]), that is there exists a prime filtration $\mathcal{F}$ of M such that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$, we have the following:

Lemma 1.9. *Let M be a finitely generated multigraded S-module. If M is almost clean and $x_k \in S$ is regular on M, then*

$$\operatorname{sdepth} M/x_kM \geq \operatorname{sdepth} M - 1.$$

Proof. Suppose that $\mathcal{F}$ is given by

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

with $M_i/M_{i-1} \cong S/P_i(-a_i)$ for some $a_i \in \mathbb{N}^n$ and some monomial prime ideals P_i. Since $\operatorname{Ass} M = \{P_1, \ldots, P_r\}$ and x_k is regular on M, we get $x_k \notin P_i$ and so x_k is regular on M_i/M_{i-1}. Set $\bar{M}_i = M_i/x_kM_i$. Then $\bar{M}_i \subset \bar{M}_{i+1}$ and $\{\bar{M}_i\}$ define a filtration $\bar{\mathcal{F}}$ of $\bar{M} = \bar{M}_r$ with $\bar{M}_i/\bar{M}_{i-1} \cong S/(P_i, x_k)(-a_i)$. Thus $\operatorname{sdepth} \bar{M} \geq \min_i \dim S/(P_i, x_k) = \operatorname{sdepth} M - 1$ (see Corollary 2.3). $\square$

The above Example 1.8 hints that if x is a regular element on M, then $\operatorname{sdepth} M/xM \leq \operatorname{sdepth} M - 1$. This is the subject of our next proposition.

Proposition 1.10. *Let M be finitely generated $\mathbb{Z}^n$-graded S-module and let x_k be regular on M. If $\mathcal{D}_1 : M/x_kM = \bigoplus_{i=1}^r \bar{m}_i K[Z_i]$, is a Stanley decomposition of M/x_kM, where $m_i \in M$ is homogeneous and $\bar{m}_i = m_i + x_kM$. Then*

$$M = \bigoplus_{i=1}^{r} m_i K[Z_i, x_k] \tag{2}$$

is a Stanley decomposition of M. In particular

$$\operatorname{sdepth} M/x_kM \leq \operatorname{sdepth} M - 1.$$

Proof. Let $N = \sum_{i=1}^r m_i K[Z_i, x_k]$. Then $N \subseteq M$. Since $\mathcal{D}_1$ is a Stanley decomposition of M/x_kM it follows that $\psi(N) = M/x_kM$ where $\psi : M \to M/x_kM$ is the canonical epimorphism. This implies that $M = x_kM + N$ as $\mathbb{Z}^n$-graded K vector spaces. We show that $M = N$. First we observe that $M = x_k^d M + N$ for all d. This follows by induction on d, because if we have $M = x_k^{d-1}M + N$, then $M = x_k^{d-1}(x_kM + N) + N = x_k^d M + x_k^{d-1}N + N = x_k^d M + N$ since $x_k^{d-1}N \subset N$. This completes the induction. Since M is finitely generated there exists an integer c such that $\deg_{x_k}(m) \geq c$ for all homogeneous elements $m \in M$. Now let $m \in M$ be a homogeneous element with $\deg_{x_k}(m) = a$ and let $d > a - c$ be an integer. Since $M = x_k^d M + N$, there exist homogeneous elements $v \in M$ and $w \in N$ such that $m = x_k^d v + w$, where $a = \deg_{x_k} v + d = \deg_{x_k} w$. It follows that $\deg_{x_k} v = a - d < c$, a contradiction. It implies that $v = 0$, hence $m = w \in N$.

Now we show that the sum $\sum_{i=1}^r m_i K[Z_i, x_k]$ is direct, that is

$$m_i K[Z_i, x_k] \cap \sum_{\substack{j=1 \\ j \neq i}}^{r} m_j K[Z_j, x_k] = (0).$$

Let $u = m_i q_i = \sum_{\substack{j=1 \\ j \neq i}}^{r} m_j q_j \in M$ be homogeneous for some q_j monomials in $K[Z_j, x_k]$ such that $\deg(u) = \deg(m_j q_j)$ for all j. Let p be the biggest power of x_k dividing q_i. If

$p=0$, then we have $\bar{u}=\bar{m}_i q_i \neq 0$ in M/x_kM since $\bar{m}_iK[Z_i]$ is a Stanley space. It follows that $\bar{u} \in \bar{m}_iK[Z_i] \cap \sum_{\substack{j=1\\ j\neq i}}^r \bar{m}_jK[Z_j]$, a contradiction. In the case of $p>0$, then in M/x_kM we get $\bar{u}=0=\sum_{\substack{j=1\\ j\neq i}}^r \bar{m}_j\bar{q}_j$. It follows that $\bar{q}_j=0$, since $\mathcal{D}_1$ is a Stanley decomposition of M/x_kM. Thus $q_j=x_kq'_j$ for some $q'_j \in K[Z_j,x_k]$ and we get $x_k(m_iq'_i-\sum_{\substack{j=1\\ j\neq i}}^r m_jq'_j)=0$, which implies $m_iq'_i-\sum_{\substack{j=1\\ j\neq i}}^r m_jq'_j=0$ since x_k is regular on M. Applying the same argument by recurrence we get $q_j=x_k^p s_j$ for some $s_j \in K[Z_j,x_k]$, and $m_is_i=\sum_{\substack{j=1\\ j\neq i}}^r m_js_j$. We set $v=m_is_i$. Since $\bar{s}_i \neq 0$, we get $\bar{v}\neq 0$ because $\bar{m}_iK[Z_i]$ is a Stanley space. On the other hand $\bar{v} \in \bar{m}_iK[Z_i] \cap \sum_{\substack{j=1\\ j\neq i}}^r \bar{m}_jK[Z_j]$. It implies that $\bar{v}=0$, a contradiction.

Finally we show that each $m_iK[Z_i,x_k]$ is a Stanley space. Indeed, suppose that $m_if=0$ for some $f\in K[Z_i,x_k]$ where $f=\sum_{j=0}^a f_jx_k^j$ such that x_k does not divide f_j for all j then $\sum_{j=0}^a m_if_jx_k^j=0$ implies that $\bar{m}_if_0=0$ in M/x_kM. We get $f_0=0$ since $\bar{m}_iK[Z_i]$ is a Stanley space. It follows that $f=x_kg$ where $g=\sum_{j=1}^a f_jx_k^{j-1}$ and from $x_km_ig=m_if=0$ we get $m_ig=0$, x_k being regular on M. Then induction on the degree of f concludes the proof since $\deg_{x_k} g<\deg_{x_k} f$. □

Corollary 1.11. *If Stanley's conjecture holds for the module M/x_iM, where $x_i\in S$ is regular on M, then it also holds for M.*

Corollary 1.12. *The equality holds in Lemma 1.7, if x_n is regular on S/I.*

The proof follows from Lemma 1.7 and Proposition 1.10 for $M=S/I$.

Corollary 1.13. *Let* $\operatorname{depth} M=t$. *If there exists $u=u_1,\ldots,u_t\in Mon(S)$ such that u is regular sequence on M then Stanley's conjecture holds for M.*

Proof. For any regular sequence $u=u_1,\ldots,u_t\in Mon(S)$ we may choose u such that $u_i=x_{i_j}$ for all $1\leq i\leq t$, where $x_{i_j}\in \operatorname{supp}(u_i)$, since for any monomial $u_i\in S$ being regular on M implies that each $x_{i_j}\in\operatorname{supp}(u_i)$ is regular on M, because if x_{i_j} belong to the set of zero divisors of M then $x_{i_j}\in P$ for some $P\in\operatorname{Ass}(M)$, so $u_i\in P$, which is not true as u_i is regular on M. Since u is a maximal regular sequence on M, we have $\operatorname{depth} M/(u_1,\ldots,u_t)M=0$. Applying Proposition 1.10 by recurrence we get $\operatorname{sdepth} M\geq \operatorname{sdepth} M/(u_1,\ldots,u_t)M+t\geq t=\operatorname{depth} M$. Hence Stanley's conjecture holds for M. □

Example 1.14. Let $S=K[x,y,z,t]$ and $M=(x,y,z)/(xy)$. Since $\operatorname{depth} M=2$ and $\{z,t\}$ is a M-regular sequence, we may apply Corollary 1.13 to see that Stanley's conjecture holds for M.

Theorem 1.15. *Let M be a finitely generated multigraded S-module. If M is almost clean and $x_k\in S$ is regular on M, then*

$$\operatorname{sdepth} M/x_kM=\operatorname{sdepth} M-1.$$

The proof follows from Lemma 1.9 and Proposition 1.10.

Theorem 1.16. *Let M be a finitely generated multigraded S-module. If M is almost clean and $u\in S$ is a monomial, which is regular on M, then* $\operatorname{sdepth} M/uM\geq\operatorname{sdepth} M-1$.

Proof. Let $u=x_{i_1}^{a_1}\ldots x_{i_t}^{a_t}$. Since u is regular on M, it follows that each $x_{i_k}\in\operatorname{supp}(u)$ is regular on M, where we denote by $\operatorname{supp}(u)$ the set of all variables x_j such that x_j divides

the monomial u. We consider an ascending chain of submodules of M between uM and M where two successive members of the chain are of the form

$$x_{i_1}^{b_1}\cdots x_{i_k}^{b_k}\cdots x_{i_t}^{b_t}M \subset x_{i_1}^{b_1}\cdots x_{i_k}^{b_k-1}\cdots x_{i_t}^{b_t}M,$$

and where $b_i \le a_i$ for all $i = 1,\ldots,t$.

We obtain

$$x_{i_1}^{b_1}\cdots x_{i_k}^{b_k-1}\cdots x_{i_t}^{b_t}M/x_{i_1}^{b_1}\cdots x_{i_k}^{b_k}\cdots x_{i_t}^{b_t}M \simeq M/x_{i_k}M,$$

since each $x_{i_k} \in \mathrm{supp}(u)$ is regular on M. Therefore Lemma 1.9 and Corollary 2.3 imply that

$$\mathrm{sdepth}(M/uM) \ge \mathrm{sdepth}(M/x_{i_k}M) = \mathrm{sdepth}\, M - 1.$$

□

2. The behavior of sdepth on short exact sequence of multigraded modules

The following "Depth Lemma" is well-known.

Lemma 2.1. [15, Lemma 1.3.9] *If*

$$0 \to U \to M \to N \to 0$$

is a short exact sequence of modules over a local ring R, then

(a) *If* $\mathrm{depth}\, M < \mathrm{depth}\, N$, *then* $\mathrm{depth}\, U = \mathrm{depth}\, M$.
(b) *If* $\mathrm{depth}\, M > \mathrm{depth}\, N$, *then* $\mathrm{depth}\, U = \mathrm{depth}\, N + 1$.

We will show that most of the statements of the "Depth Lemma" are wrong if we replace depth by sdepth. We first observe

Lemma 2.2. *Let*

$$0 \to U \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be an exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then

$$\mathrm{sdepth}\, M \ge \min\{\mathrm{sdepth}\, U, \mathrm{sdepth}\, N\}$$

Proof. Let $\mathcal{D}: \ U = \bigoplus_{i=1}^r u_iK[Z_i]$ be a Stanley decomposition of U with $\mathrm{sdepth}(\mathcal{D}) = \mathrm{sdepth}\, U$ and let $\mathcal{D}': \ N = \bigoplus_{j=1}^s n_jK[Z_j']$ be a Stanley decomposition of N with $\mathrm{sdepth}(\mathcal{D}')$ $= \mathrm{sdepth}\, N$. Since f is injective map, we may suppose that f is an inclusion. Let $n_j' \in M$ be a $\mathbb{Z}^n$ homogeneous element such that $g(n_j') = n_j$. Clearly, $M = \sum_{i=1}^r u_iK[Z_i] + \sum_{j=1}^s n_j'K[Z_j']$. We prove that the sum $\sum_{i=1}^r u_iK[Z_i] + \sum_{j=1}^s n_j'K[Z_j']$ is direct. Set $V = \sum_j n_j'K[Z_j']$. Since the exact sequence splits as linear spaces we see that $U \cap V = \{0\}$. Clearly $\mathcal{D}$ is already a Stanley decomposition of U and remains to show only that if $y \in n_j'K[Z_j'] \cap \sum_{\substack{k=1\\k\neq j}}^s n_k'K[Z_k']$ then $y = 0$. As $g(y) \in n_jK[Z_j'] \cap \sum_{\substack{k=1\\k\neq j}}^s n_kK[Z_k'] = \{0\}$, we see that $y \in U$, that is $y \in U \cap V = \{0\}$. □

Corollary 2.3. *Let*

$$(0) = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = M$$

be an ascending chain of $\mathbb{Z}^n$-graded submodules of M. Then

$$\mathrm{sdepth}\, M \ge \min\{\mathrm{sdepth}\, M_i/M_{i-1} : \ i \in \{1,\ldots,r\}\} \tag{3}$$

for all $i \in [r]$.

Proof. We consider the exact sequence of $\mathbb{Z}^n$-graded submodules of M such that

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0.$$

By Lemma 2.2, we get $\operatorname{sdepth} M_i \geq \min\{\operatorname{sdepth} M_{i-1}, \operatorname{sdepth} M_i/M_{i-1}\}$. We apply induction to prove the inequality (3). For $i = 1$ this holds clearly. We suppose (3) is true for $i = t$ then we have

$$\operatorname{sdepth} M_t \geq \min\{\operatorname{sdepth} M_i/M_{i-1} : \ i \in \{1, \dots, t\}\}.$$

Let $i = t + 1$ then we have $\operatorname{sdepth} M_{t+1} \geq \min\{\operatorname{sdepth} M_t, \operatorname{sdepth} M_{t+1}/M_t\}$, which is enough. $\square$

The analogue of Lemma 2.1(a) only holds under an additional assumption.

Corollary 2.4. *In the hypothesis of Lemma 2.2 suppose that* $\operatorname{sdepth} M < \operatorname{sdepth} N$. *Then* $\operatorname{sdepth} M \geq \operatorname{sdepth} U$.

Proof. If $\operatorname{sdepth} M < \operatorname{sdepth} U$, we get $\operatorname{sdepth} M < \min\{\operatorname{sdepth} U, \operatorname{sdepth} N\}$ contradicting Lemma 2.2. $\square$

The analogue of 2.1(b) is wrong.

Example 2.5. Let $S = K[x, y, z]$, $M = (x, y, z)$. In the exact sequence $0 \to M \to S \to K \to 0$, we have $\operatorname{sdepth} S = 3 > \operatorname{sdepth} K = 0$ but $\operatorname{sdepth} M = 2 \neq \operatorname{sdepth} K + 1$.

Note that the case treated in Proposition 1.10, that is the short exact sequence $0 \to M \xrightarrow{x_k} M \to M/x_kM \to 0$, and Lemma 2.7 apparently hints that some analogue of (b) from "Depth Lemma" in the frame of sdepth might be true. Unfortunately, this is not the case as shows the following:

Example 2.6. We have a resolution $0 \to \Omega^1 m \to S^3 \to m \to 0$, where $S = K[x, y, z]$ and $m = (x, y, z)$. Then $\Omega^1 m$ is not free because otherwise $\operatorname{proj\,dim}_S m$ should be 1, which is not true. If $\operatorname{sdepth} \Omega^1 m = 3$ then follows $\Omega^1 m$ free by the elementary Lemma 2.9. Thus $\operatorname{sdepth} \Omega^1 m \leq 2 = \operatorname{sdepth} m$.

However it remains still the problem in general that if for an exact sequence $0 \to U \to M \to N \to 0$, $\operatorname{sdepth} M > \operatorname{sdepth} N$ implies $\operatorname{sdepth} U \geq \operatorname{sdepth} N + 1$. In general this is false (see Example 2.6) but we prove this result in a special case.

Lemma 2.7. *If* $I \subset S = K[x_1, \dots, x_n]$ *is a monomial complete intersection, then* $\operatorname{sdepth} I \geq \operatorname{sdepth} S/I + 1$.

Proof. Let $\{v_1, \dots, v_m\}$ be the regular sequence of monomials generating I. Since $\operatorname{sdepth} S/I = n - m$, by applying [13, Theorem 1.1] recursively, and $\operatorname{sdepth} I \geq n - m + 1$, by [7], or [9, Proposition 3.4], it follows the desired result. $\square$

In general for any monomial ideal the inequality in above lemma is still an open question. This inequality motivates that $\operatorname{sdepth} I \geq \operatorname{sdepth} J/I + 1$ for any two monomial ideals $I \subset J \subset S$. But this inequality does not hold as shows the following example:

Example 2.8. *Let* $S = K[x, y]$, $I = (xy, y^2)$, $J = I + (x^2)$. *Then we have* $\operatorname{sdepth} J/I = 1 = \operatorname{sdepth} I = \operatorname{sdepth} J$.

Lemma 2.9. *If* M *is multigraded* S*-module,* $S = K[x_1, \dots, x_n]$ *with* $\operatorname{sdepth} M = n$ *then* M *is free.*

Proof. If $\operatorname{sdepth} M = n$, then we have a Stanley decomposition of the form $M = \oplus_i u_i S$ and $u_i S$ are free S-modules. The direct sum is of linear spaces but it turns out to be of free S-modules. $\square$

3. The behavior of sdepth on algebra tensor product

Theorem 3.1. *Let $I \subset S_1 = K[x_1, \dots, x_n]$, $J \subset S_2 = K[y_1, \dots, y_m]$ be monomial ideals and $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Then* $\operatorname{sdepth} S_1/I + \operatorname{sdepth} S_2/J \leq \operatorname{sdepth} S/(IS, JS)$.

Proof. Let

$$\mathcal{D}_1 : \ S_1/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

be a Stanley decomposition of S_1/I such that $\operatorname{sdepth} \mathcal{D}_1 = \operatorname{sdepth} S_1/I$ and

$$\mathcal{D}_2 : \ S_2/J = \bigoplus_{j=1}^{s} v_j K[W_j]$$

be a Stanley decomposition of S_2/J such that $\operatorname{sdepth} \mathcal{D}_2 = \operatorname{sdepth} S_2/J$. Then we have

$$\begin{aligned} S/IS &= S_1[y_1, \dots, y_m]/IS \\ &= (S_1/I)[y_1, \dots, y_m] \\ &= \bigoplus_{i=1}^{r} u_i K[Z_i][y_1, \dots, y_m] \\ &= \bigoplus_{i=1}^{r} u_i K[Z_i, y_1, \dots, y_m] \end{aligned}$$

and

$$\begin{aligned} S/JS &= S_2[x_1, \dots, x_n]/JS \\ &= (S_2/J)[x_1, \dots, x_n] \\ &= \bigoplus_{j=1}^{s} v_j K[W_j][x_1, \dots, x_n] \\ &= \bigoplus_{j=1}^{s} v_j K[W_j, x_1, \dots, x_n]. \end{aligned}$$

We claim that

$$S/(IS, JS) = \bigoplus_{i,j} u_i v_j K[Z_i, W_j].$$

Let $w \in (IS, JS)^c = S/(IS, JS)$ be a monomial; that is, $w \in S$ and $w \notin (IS, JS)$. We have $w \notin IS$ and $w \notin JS$. It follows that $w \in (IS)^c$ and $w \in (JS)^c$. Hence there exist i and j such that $w \in u_i K[Z_i, y_1, \dots, y_m]$ and $w \in v_j K[W_j, x_1, \dots, x_n]$. So we have $w \in u_i K[Z_i, y_1, \dots, y_m] \cap v_j K[W_j, x_1, \dots, x_n]$ and $u_i K[Z_i, y_1, \dots, y_m] \cap v_j K[W_j, x_1, \dots, x_n] = u_i v_j K[Z_i, W_j]$, since $u_i \in S_1$ and $v_j \in S_2$.

In order to prove the opposite inclusion, consider a monomial $v \in u_i v_j K[Z_i, W_j]$. Then $v \in u_i K[Z_i, y_1, \dots, y_n] \subset (IS)^c$ and similarly $v \in (JS)^c$. Thus $v \in (IS, JS)^c$. So $S/(IS, JS) = \sum_{i,j} u_i v_j K[Z_i, W_j]$.

Now we prove that this sum is direct. Let i_1, $i_2 \in [r]$ and j_1, $j_2 \in [s]$ be such that $(i_1, j_1) \neq (i_2, j_2)$, let us say $i_1 \neq i_2$. Then $u_{i_1} v_{j_1} K[Z_{i_1}, W_{j_1}] \cap u_{i_2} v_{j_2} K[Z_{i_2}, W_{j_2}] \subset u_{i_1} K[Z_{i_1}, y_1, \dots, y_m] \cap u_{i_2} K[Z_{i_2}, y_1, \dots, y_m] = \{0\}$, which shows our claim. It follows that $\operatorname{sdepth} S_1/I + \operatorname{sdepth} S_2/J \leq \operatorname{sdepth} S/(IS, JS)$. $\square$

The following example shows that the inequality in the above theorem can be strict.

Example 3.2. Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ be a polynomial ring over the field K. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subset S_1 = K[x_1, x_2, x_3, x_4]$ be the ideal of the polynomial ring S_1 and $J = (x_5x_7, x_5x_8, x_6x_7, x_6x_8) \subset S_2 = K[x_5, x_6, x_7, x_8]$ be the ideal of the polynomial ring S_2. Consider the ideal $(IS, JS) \subset S$, then a Stanley decomposition $\mathcal{D}$ of $S/(IS, JS)$ is
$\mathcal{D}:\ S/(IS, JS) = K[x_1, x_2, x_5] \oplus x_3K[x_3, x_5, x_6] \oplus x_4K[x_4, x_5, x_6] \oplus x_6K[x_1, x_2, x_6] \oplus x_7K[x_1, x_2, x_7] \oplus x_8K[x_1, x_2, x_8] \oplus x_3x_4K[x_3, x_4, x_5] \oplus x_3x_7K[x_3, x_7, x_8] \oplus x_3x_8K[x_3, x_4, x_8] \oplus x_4x_7K[x_3, x_4, x_7] \oplus x_4x_8K[x_4, x_7, x_8] \oplus x_5x_6K[x_1, x_5, x_6] \oplus x_7x_8K[x_1, x_7, x_8]$
$\oplus\, x_2x_5x_6K[x_1, x_2, x_5, x_6] \oplus x_3x_4x_6K[x_3, x_4, x_5, x_6] \oplus x_2x_7x_8K[x_1, x_2, x_7, x_8]$
$\oplus x_3x_4x_7x_8K[x_3, x_4, x_7, x_8]$, hence $\operatorname{sdepth} \mathcal{D} = 3$. Note that $\operatorname{sdepth} S_1/I = 1$ with a Stanley decomposition $S_1/I = K[x_1, x_2] \oplus x_3K[x_3] \oplus x_4K[x_3, x_4]$. We observe that $\operatorname{sdepth} S_1/I$ can not be greater than one. Similarly we have $\operatorname{sdepth} S_2/J = 1$. Hence we obtain that $\operatorname{sdepth} S_1/I + \operatorname{sdepth} S_2/J < \operatorname{sdepth} S/(IS, JS)$.

The following corollary is a particular case of [13, Theorem 1.1].

Corollary 3.3. *Let $I \subset S$ be a monomial ideal and $u \in S$ is a monomial, which is regular on S/I. Then* $\operatorname{sdepth} S/(I, u) \geq \operatorname{sdepth} S/I - 1$.

Proof. Renumbering $x_i \in \operatorname{supp}(u_j)$ for all $u_j \in G(I)$, we may suppose that I is generated by a monomial ideal $J \subset S_1 = K[x_1, \ldots, x_r]$ and $u \in S_2 = K[x_{r+1}, \ldots, x_n]$ for some $1 < r < n$. Then $\operatorname{sdepth} S/(I, u) \geq \operatorname{sdepth} S_1/J + \operatorname{sdepth} S_2/(u)$, by Theorem 3.1. Since $\operatorname{sdepth} S_2/(u) = n - r - 1$ and $\operatorname{sdepth} S/I = \operatorname{sdepth} S_1/J + n - r$, by [13, Lemma 1.2], it follows that $\operatorname{sdepth} S/(I, u) \geq \operatorname{sdepth} S/I - 1$. $\square$

In the analogue of Theorem 3.1 for depth we have equality, that is
$\operatorname{depth} S/(IS, JS) = \operatorname{depth} S_1/I + \operatorname{depth} S_2/J$ (see [15, Theorem 2.2.21]). We note that if Stanley's conjecture hold for the modules S_1/I and S_2/J it holds also for $S/(IS, JS)$.

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Asia Rauf, Abdus Salam School of Mathematical Sciences, GC University, Lahore
E-mail address: asia.rauf@gmail.com